I Want to Break Square-free: The $4p - 1$ Factorization Method and Its RSA Backdoor Viability

SECRYPT 2019

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Overview

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  - interesting as a backdoor

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- we have $E(\mathbb{Z}_N) \cong E(\mathbb{F}_p) \oplus E(\mathbb{F}_q)$
- if $|E(\mathbb{F}_p)| = p$, multiplication by N annihilates the first summand, which reveals p

The basic idea

For any $k \in \mathbb{Z}$, there are division polynomials $\psi_k, \phi_k, \omega_k \in \mathbb{Z}[x]$:

$$k \cdot P = \left(\frac{\phi_k(x)}{\psi_k^2(x)}, \frac{\omega_k(x,y)}{\psi_k^3(x,y)} \right)$$

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- thus we can recover $p = \gcd(N, \psi_N(x))$

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- thus if $4p - 1 = Ds^2$ and $H_{-D}(j(E)) \equiv 0 \pmod{p}$, then $|E(\mathbb{F}_p)| = p$ in one half of cases

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7. if the computation of $N \cdot P$ fails, compute a factor of N as $\gcd(\psi_N(x), N)$.

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- current record: $D \approx 2^{53}$

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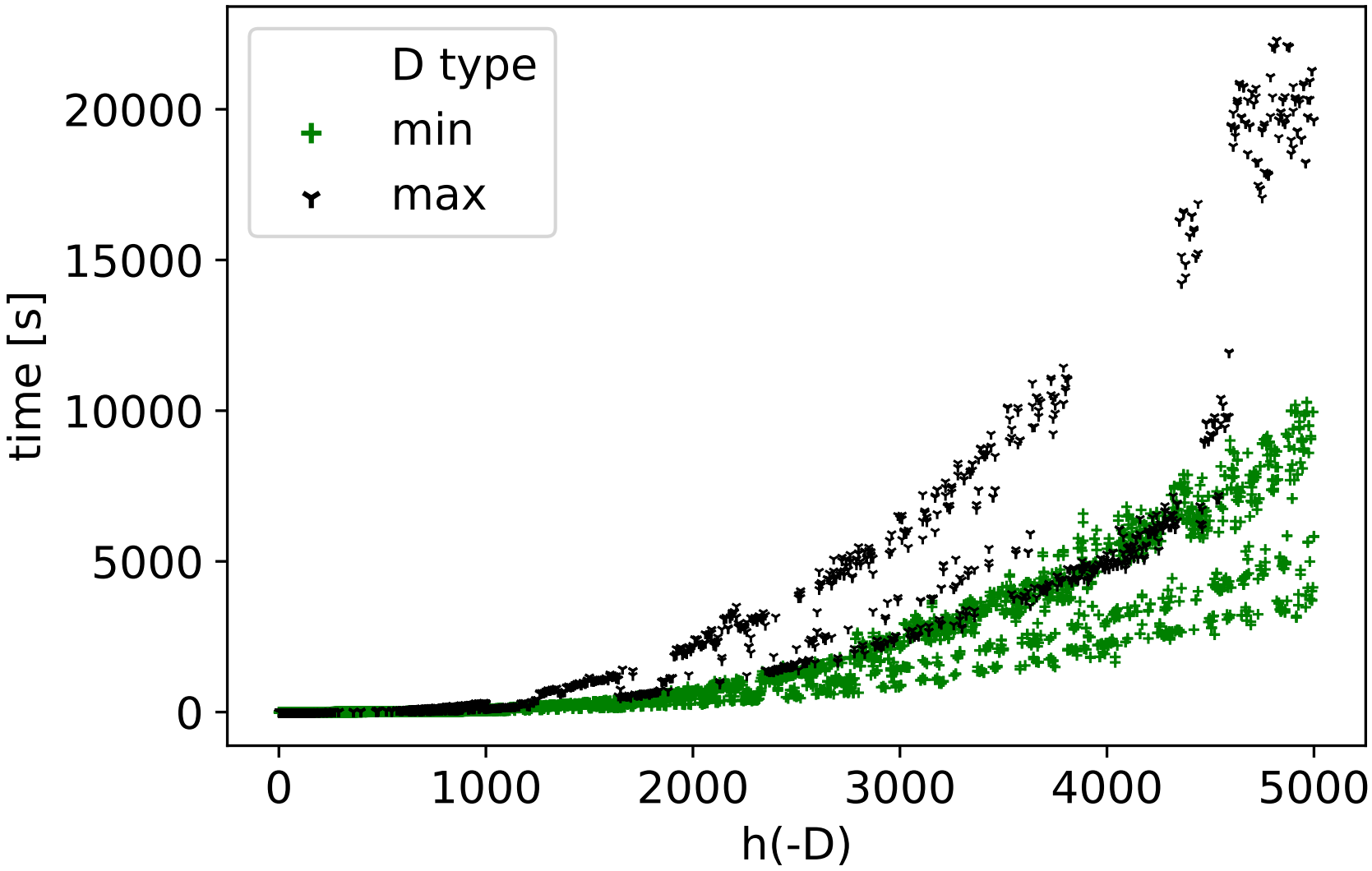
D type
min
max
20000
15000
10000
5000
0
time [s]
0
1000
2000
3000
4000
5000
h(-D)

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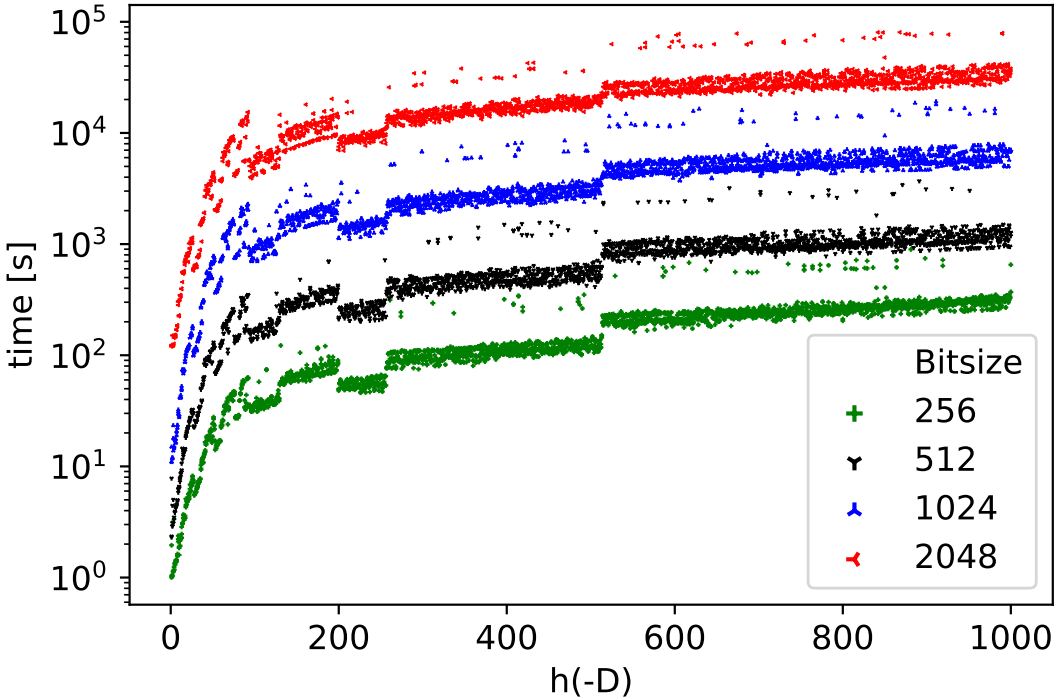

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- only $\frac{1}{\sqrt{X}}$ of primes $p < X$ satisfy it
- still could serve as an interesting backdoor (e.g., on black-box devices)
- generation of vulnerable primes for given D is easy

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- if D is leaked, anyone can perform the factorization

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3. many private keys - batch GCD reveals the backdoor if D is not unique per keypair

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Thank you for your attention.

All data and implementation are publicly available at `https://crocs.fi.muni.cz/public/papers/Secrypt2019`.